

Theorem 4 (Poole 3.12): Let A be a $n \times n$ matrix. The following statements are equivalent (they are either all TRUE statements or all FALSE statements).

1. The matrix A is invertible.
2. $\text{rank}(A) = n$. ✓
3. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every \mathbf{b} in \mathbb{R}^n .
4. The homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.
5. The reduced row echelon form of A is I_n .

Note 4: Poole calls theorem 4 the **Fundamental Theorem of Invertible Matrices**.

Example 5: Is the matrix $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$ invertible? Explain.

$$\begin{array}{l}
 A \quad R_4 := R_4 - R_1 \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} R_4 := R_4 + 2R_2 \\ R_3 := R_3 - R_2 \end{array} \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} \quad R_4 := R_4 + 3R_3 \\
 \\
 \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad \underline{\text{rank}(A) = 4}
 \end{array}$$

By the fundamental theorem of invertible matrices (FTIM),
The matrix A is invertible